DYNAMICAL TUNNELING IN OPTICAL CAVITIES

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ABSTRACT

The lifetime of whispering gallery modes in a dielectric cavity with a metallic inclusion is shown to fluctuate by orders of magnitude when size and location of the inclusion are varied. We ascribe these fluctuations to tunneling transitions between resonances quantized in different regions of *phase space*. This interpretation is confirmed by a comparison of the classical phase space structure with the Husimi distribution of the resonant modes. A model Hamiltonian is introduced that describes the phenomenon and shows that it can be expected in a more general class of systems.

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In recent years a lot of experimental effort [1] has been devoted to dielectric microresonators. In these resonators long-lived whispering gallery (WG) modes are created by total internal reflection of light circulating just inside the surface of the dielectric. WG modes can have Q values exceeding [2] 10^9 and they are of great interest for applications such as microlasers and optical interconnects. For optical cavities that are deformed from rotationally symmetric shape it was demonstrated [3] that the emission pattern of WG modes can be understood in a ray-optics picture. The ray dynamics typically exhibits both stable and chaotic trajectories, and it was argued that the presence of chaos should lead to a broadening of the WG modes with increasing deformation.

In this paper we study effects in dielectric cavities that are beyond a ray-optics model. We show that tunneling transitions between classically disconnected regions in phase space can lead to fluctuations in the lifetime of WG modes by several orders of magnitude. This is demonstrated numerically by solving the wave equation for a dielectric that has a permeable coating and the shape of the annular billiard. The fluctuations arise from avoided crossings of WG modes with much broader resonances of the dielectric. The coating serves to make this effect more pronounced by raising all lifetimes above those obtained from total internal reflection alone. The Husimi projections of the relevant modes reveal that the broad resonances and the WG modes

are localized in chaotic and regular regions of phase space, respectively. This motivates us to model the dielectric in terms of a tunneling Hamiltonian where states quantized in different regions of phase space are connected by small tunneling matrix elements. We derive the quantum scattering matrix associated with the tunneling Hamiltonian and obtain the widths of the quasibound states. In agreement with our numerical observation we find strong fluctuations in the lifetime of WG modes close to avoided crossings with modes localized in a different region of phase space.

The idea that disconneted regions in classical phase space can give rise to quantum tunneling has previously been introduced by Davis and Heller [4], who dubbed this notion dynamical tunneling. Dynamical tunneling has been observed [5, 6, 7, 8] in the spectrum of *closed* quantum systems whose classical counterparts exhibit both regular and chaotic motion. In such systems dynamical tunneling leads to statistical fluctuations in the splittings of nearly degenerate energy levels. While the resulting level splittings are typically so small that their experimental detection seems unlikely, we believe that the tunneling effects giving rise to fluctuations in resonance lifetimes of *open* systems are amenable to experimental verification.

We numerically study a dielectric cavity with the shape of the annular billiard [6, 8]. The system [Fig. 1 a)] consists of a cylindrical dielectric with radius R and index of refraction $n > 1$ that is suspended in air. Embedded in this cylinder and off-center by an amount δ is a totally reflecting (metallic) rod of radius $a < R - \delta$. The surface of the dielectric is covered by a penetrable metallic film. The dielectric function of the system is written as $\epsilon = \epsilon^{\mathrm{bulk}} + \epsilon^{\mathrm{film}}$ where $\epsilon^{\mathrm{bulk}} = n^2$ inside the dielectric and $\epsilon^{\mathrm{bulk}} = 1$ outside. The metallic film is taken as a δ-function layer, $\epsilon^{\mathrm{film}} = -\eta\, n^2 R\, \delta(r - R)$ with $\eta > 0$, corresponding to a purely imaginary refractive index. This neglects any phase shifts in the film (caused by a real part of the index), as well as the frequency-dependence of the absorption. Absorption by the inclusion is neglected because we expect the dominant loss to be caused by the outer interface.

We solve for the quasibound states of the annular billiard using wave function matching. The electric field Ψ when polarized parallel to the cylinder axis satisfies the scalar wave equation

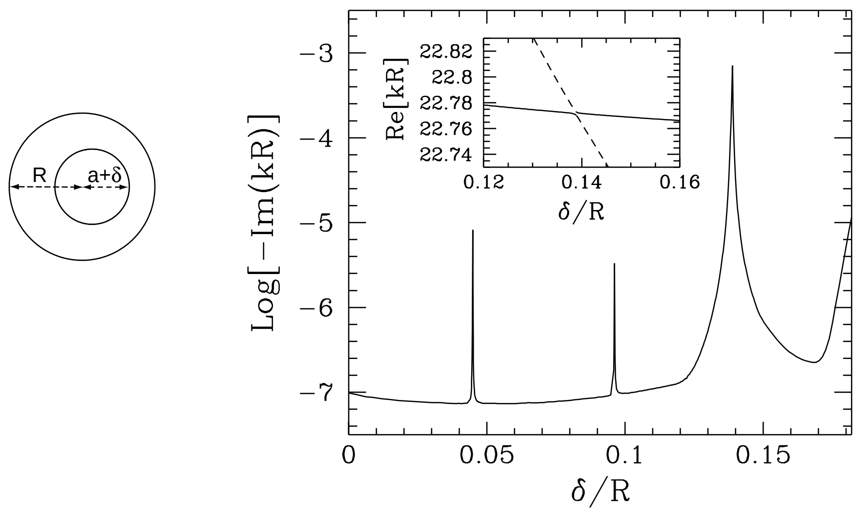
$$\nabla^2\Psi + \epsilon(\mathbf{r})k^2\Psi = 0, \tag{1}$$

and is continuous at $r = R$. The δ-function in ϵ creates a jump in the derivative, $\partial\Psi/\partial r|_R = \eta\, n^2\, k^2 R\, \Psi(R)$. We search for a solution *with no incoming waves*,

$$\Psi_< = \sum_\mu A_\mu \Psi^+_\mu, \quad \Psi_> = \sum_\mu B_\mu H^{(1)}_\mu(kr)\cos\mu\phi, \tag{2}$$

$$\Psi^+ = H^{(2)}_\mu(nkr)\cos\mu\phi + \sum_\nu S^{(I)}_{\nu\mu} H^{(1)}_\nu(nkr)\cos\nu\phi, \tag{3}$$

where $\Psi_<$ and $\Psi_>$ denote the electric field for $r < R$ and $r > R$, respectively. We use polar coordinates r, ϕ with origin in the center of the dielectric, and $H^{(1,2)}_\mu$ are the Hankel functions of first and second kind. The scattering matrix $S^{(I)}$ of the inner annulus is known explicitly [8]. Matching Ψ and $\partial\Psi/\partial r$ at the surface $r = R$, one



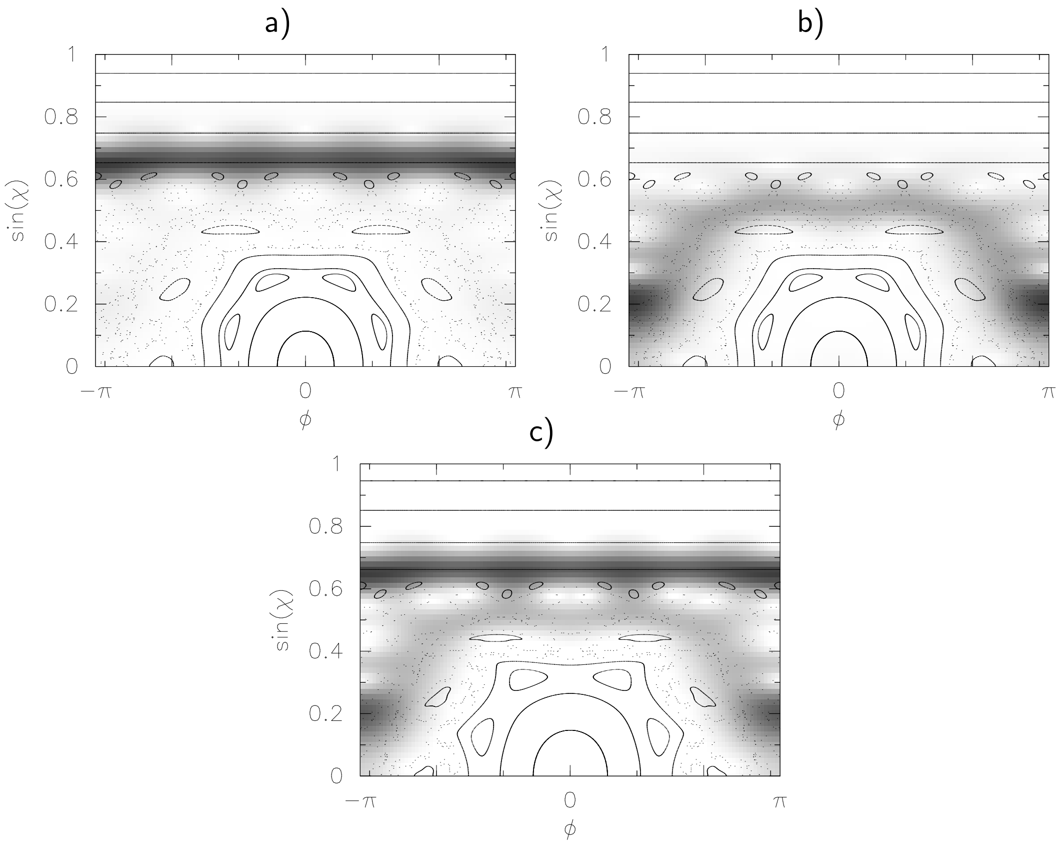
a) Cross seection of cylindrical resonator with off-center metallic inclusion (inner disk). b) Width of the WG mode (2,30) vs. eccentricity δ/R. Inset: Positions of WG mode (2,30) (solid line) and the closest broad resonance (dashed).

obtains a set of simultaneous equations for the Fourier coefficients A_μ, B_μ, which can only be solved at discrete complex values of the wavenumber k. To each solution we assign the radial and angular momentum quantum numbers (u, m) of the state in the concentric billiard ($\delta = 0$) from which it evolved.

The *closed* annular billiard ($\eta \to \infty$) exhibits mixed classical dynamics [6]. Let χ be the *angle of incidence* measured from the surface normal. Then WG trajectories are those which never hit the inner cylinder, i.e. their impact parameter is $R\,|\sin\chi| > a{+}\delta$. Their motion is regular, with a *conserved* χ. The phase space for $R\,|\sin\chi| < a{+}\delta$ contains both regular and chaotic regions if $\delta > 0$. For chaotic rays, χ fluctuates between reflections and can become very small. In the *open* system, this means that modes supported by the chaotic domain have widths that are several orders of magnitude broader than the widths of WG modes, because the outer coating is far more penetrable close to normal incidence than at the grazing angles allowed for WG rays.

Fig. 1 b) shows the width of the even WG mode (2,30) corresponding to $\sin\chi = 0.66$ (at $\delta = 0$) as a function of δ/R. We chose $n = 2$, $\eta = 1$ and kept $a + \delta = 0.65$ fixed. This ensures that WG modes with $|\sin\chi| > (a+\delta)/R$ remain classically undisturbed. The width is seen to fluctuate by orders of magnitude. This is caused by avoided crossings with broader resonances, as we show in the inset for the large peak at $\delta/R \approx 0.139$ [9]. Here, the dashed line corresponds to a broad resonance of width $-\mathrm{Im}(kR) \approx 10^{-3}$.

We now want to associate these resonances with different regions of the clas-



The projection ρ_{SOS} of the Husimi function onto the Poincaré surface of section, superimposed with classical trajectories to indicate the extend of regular and chaotic regions. The uncertainty in ϕ is 0.5 radians; $\delta/R = 0.135$ in a) and b), with a) showing the phase space density of the WG mode and b) that of the broad resonance. In c), ρ_{SOS} is shown for the WG state at $\delta/R = 0.139$.

sical phase space spanned by r, ϕ and their conjugate momenta (p_r, p_ϕ). This is achieved using the Husimi function, i.e. the overlap of $\Psi_<$ with a minimum-uncertainty wavepacket Φ centered at $\bar{r}, \bar{\phi}, \bar{p}_r, \bar{p}_\phi$. The resulting phase space density $\rho_H(\bar{r}, \bar{\phi}, \bar{p}_r, \bar{p}_\phi)$ characterizes $\Psi_<$. To obtain a classical phase-space portrait, we employ a Poincaré surface of section with ϕ and $\sin\chi$ as coordinates. By semiclassical arguments, $\sin\chi$ can be directly related to p_ϕ. One can then project ρ_H onto the surface of section by integrating out $\bar{p}_r$, choosing the spread of Φ around $\bar{r}$ to be zero and setting $\bar{r} = R$.

The resulting density distribution ρ_{SOS} is shown in Fig. 2 for the two resonances singled out in the inset to Fig. 1 b). Away from the avoided crossing, the WG mode is localized in the regular regime at $\sin\chi \approx 0.66$, whereas the broad resonance is supported in the chaotic domain. However, close to the anticrossing at $\delta/R \approx 0.139$, ρ_{SOS} [Fig. 2 c)] reflects a strong coupling between the regular and chaotic parts of phase space due to interference of the two resonances. In contrast to Figs. 2 a) and b), this is a *classically forbidden* phase space distribution made possible

only by dynamical tunneling. It causes the lifetime of the WG mode to approach that of the chaotic state. We find fluctuations similar to Fig. 1 b) for all WG modes with $|\sin\chi| > (a+\delta)/R$.

Our analytical approach is motivated by the observation that the modes of the cavity have overlap with distinct regions of the classical phase space. Exploiting the analogy between the two-dimensional wave equation and the two-dimensional Schrödinger equation we model the system by the following Hamiltonian

$$H = H_0 + H_T. \tag{4}$$

Here, H_0 describes states quantized in two classically disconnected regions in phase space as well as the channel region when these subsystems are *totally disconnected* from each other. The couplings between the subregions are described by H_T. Written in a basis where the various subsystems are diagonal, H_0 has the form

$$H_0 = \sum_i \mathcal{E}_i q_i^\dagger q_i + \sum_\mu E_\mu c_\mu^\dagger c_\mu + \sum_a \int dE E d_{aE}^\dagger d_{aE}. \tag{5}$$

Here, the creation (annihilation) operators $q_i^\dagger$ and $c_\mu^\dagger$ (q_i and c_μ) arise from the quantization of the two respective regions of phase space. The corresponding quantum states will be referred to as regular and chaotic states below. A generalization of H_0 to include more regions in phase space is straightforward. There is a continuum of channel states with corresponding operators $d_{aE}^\dagger$, d_{aE}, where $a = 1, 2, \ldots, M$ denotes the channels. The tunneling Hamiltonian H_T is given by

$$\begin{aligned} H_T = \quad & \left(\sum_{ai} \int dE U_{ai}(E) d_{aE}^\dagger q_i + H.c.\right) + \\ & + \left(\sum_{i\mu} V_{i\mu} q_i^\dagger c_\mu + H.c.\right) + \left(\sum_{a\mu} \int dE W_{a\mu}(E) d_{aE}^\dagger c_\mu + H.c.\right). \end{aligned} \tag{6}$$

It couples both the regular states and the chaotic states directly to the continuum. Moreover, there is a coupling between the regular and the chaotic states via the matrix elements $V_{i\mu}$.

The Hamiltonian (4) previously has been investigated in various limiting cases. The case $V_{i\mu} = U_{ai} = 0$ together with the assumption that the matrix elements $W_{a\mu}$ be random variables defines a model for chaotic scattering which has been the subject of intensive research in the past decade [10, 11]. The case of vanishing channel couplings, $U_{ai} = W_{a\mu} = 0$ has been studied in recent work [4-6] on the splitting distribution in *closed* quantum systems. Here, we shall investigate the effects of tunneling transitions between the regular and the chaotic region ($V_{i\mu} \neq 0$) when these regions are weakly coupled to the channels. The channel couplings are assumed so weak that the typical widths of both the regular and the chaotic resonances are smaller than their respective energy spacing (the regime of isolated resonances).

We calculate the scattering matrix $S_{ab}(E) = \delta_{ab} - 2\pi i T_{ab}$ at energy E from the Lippmann-Schwinger equation

$$T = H_T + H_T \frac{1}{E - H_0 + i\eta} T \tag{7}$$

for the transition operator T (η is positive infinitesimal). This equation is solved by the explicit summation of the associated Born series [12]. Here it is sufficient to discuss the solution in the case that the direct coupling between regular and channel states is much weaker than the coupling arising from indirect processes involving chaotic states. One finds that the S matrix has the form $S_{ab} = \delta_{ab} - 2\pi i T^C_{ab} - 2\pi i T^R_{ab}$ where T^C_{ab} comprises all scattering processes involving only chaotic states while all remaining processes are included in T^R_{ab}. The poles of T^R_{ab} are located at complex energies $E_i = \epsilon_i - (i/2)\Gamma_i$, where ϵ_i is the position and Γ_i the width of the i-th resonance. Assuming that this resonance is isolated from all other resonances, the result is

$$\Gamma_i = 2\pi \sum_a \left| \sum_\mu \frac{W_{a\mu} V^*_{i\mu}}{\epsilon_i - E_\mu + i\pi \sum_b W^*_{b\mu} W_{b\mu}} \right|^2 , \tag{8}$$

where the matrix elements are evaluated at energy ϵ_i. This resembles the expression for the transition rate in second-order perturbation theory, but in contrast to the perturbative expression the denominator in Eq. (8) has a finite imaginary part. In accordance with our numerical results for the lifetime of WG modes, Eq. (8) shows a large increase of the width whenever a regular WG mode (with index i) crosses a chaotic mode (with index μ). In this way, multi-step tunneling processes involving chaotic states cause strong fluctuations in the resonance widths.

It is known [13] that the interference of resonances may strongly affect their positions and widths. To study this effect very close to an avoided crossing we consider the nonhermitian Hamiltonian

$$H = \begin{pmatrix} \mathcal{E}_i & V \\ V^* & E_\mu \end{pmatrix} - i \begin{pmatrix} 0 & 0 \\ 0 & \bar{\Gamma}/2 \end{pmatrix} \tag{9}$$

as a simple model of two resonances that are coupled to single open channel. If the separation $|\mathcal{E}_i - E_\mu|$ is much larger than $|V|$, there are two eigenstates with width close to 0 and $\bar{\Gamma}$, respectively. For small values of $|\mathcal{E}_i - E_\mu|$, one finds an avoided crossing of the resonance positions (the real parts of the eigenvalues). At the same time, both widths approach the value $\bar{\Gamma}/2$. We expect the sharp resonance typically to aquire a width of order $\bar{\Gamma}$ (where $\bar{\Gamma}$ is the width of the broader resonance when both resonances are separated) even in the general case of arbitrary number of open channels. This is clearly reflected in our numerical data presented in Fig. 1 b).

Further progress can be made assuming that the quantum states localized in the chaotic regions of phase space obey random-matrix behavior. In this case, the width (8) becomes a statistical quantity and one can compute its probability

distribution $P(\Gamma_i)$ over a large number of anticrossings. Due to avoided crossings, this distribution shows a power-law decay $P(\Gamma_i) \sim \Gamma_i^{-3/2}$ in the regime $\Gamma_i \ll \bar{\Gamma}$, where $\bar{\Gamma}$ denotes the average width of the chaotic resonances. The derivation of the distribution $P(\Gamma_i)$ and a further discussion is defered to a forthcomming publication [14].

In summary, we have studied the lifetime and the phase space distribution of WG modes in a dielectric cavity. Dynamical tunneling leads to large fluctuations of the lifetimes as a function of the asymmetry parameter, as we demonstrated numerically for the open annular billiard. This system can be realized using dielectric microcavities.

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